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THE PROBLEM OF THE STABILITY OF THE EQUILIBRIUM POSITION OF A HAMILTONIAN SYSTEM AT 3:1 RESONANCE†

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The stability of the equilibrium position of an autonomous Hamiltonian system with two degrees of freedom is investigated. It is assumed that the equilibrium is stable in the linear approximation, the frequencies ω_1 and ω_2 of small oscillations are connected by the resonance relation $\omega_1 = 3\omega_2$, and the Hamilton function is not sign-definite in the neighbourhood of the equilibrium position. The critical case when it is necessary to take into account terms higher than the fourth power in the expansion of the Hamilton function in series in order to obtain strictly valid conclusions on the stability of the equilibrium position is investigated. Sufficient conditions for stability and instability, which are expressed in terms of the coefficients of the expansion up to the sixth power inclusive, are obtained. The results are used in the problem of the stability of steady rotation of a dynamically symmetrical artificial satellite – of a rigid body around the normal to the plane of the circular orbit of its centre of mass.

1. FORMULATION OF THE PROBLEM

Consider an autonomous Hamiltonian system with two degrees of freedom. Suppose the origin of coordinates $q_j = 0, p_j = 0$ $(j = 1, 2)$ of phase space is an equilibrium position of the system, while the Hamilton function $H(q_1, q_2, p_1, p_2)$ is analytical in a certain neighbourhood of it. If the function H is sign-definite, then by Lyapunov's theorem, the equilibrium position is stable [1] (we can take the function H as Lyapunov's function). We will assume that the function H is not sign-definite but the eigenvalues $\pm i\omega_1$, $\pm i\omega_2$, of the matrix of the linearized equations of the perturbed motion are pure imaginary and different, so that the equilibrium is stable in the linear approximation.

We will assume that forth-order resonance occurs in the system, i.e. the frequencies ω_1 and ω_2 of small oscillations are connected by the relation $\omega_1 = 3\omega_2$. With a suitable choice of the canonically conjugate variables q_j, p_j we can write the Hamilton function [2] in the form

$$H = \omega_1 r_1 - \omega_2 r_2 + c_{20} r_1^2 + c_{11} r_1 r_2 + c_{02} r_2^2 + r_1^{1/2} r_2^{3/2}(a_{13}\sin\phi + b_{13}\cos\phi) + O_6 \tag{1.1}$$

$$\phi = \varphi_1 + 3\varphi_2, \quad q_j = \sqrt{2r_j}\sin\varphi_j, \quad p_j = \sqrt{2r_j}\cos\varphi_j, \quad j = 1, 2$$

where $c_{20}, c_{11}, c_{02}, a_{13}$ and b_{13} are constant coefficients, and the set of terms, the power of which is higher than the fifth in q_j, p_j is denoted by O_6.

Suppose resonance terms are actually present in expansion (1.1), i.e. $a_{13}^2 + b_{13}^2 \neq 0$. We will put

$$\varkappa = |c_{20} + 3c_{11} + 9c_{02}|\left[27(a_{13}^2 + b_{13}^2)\right]^{-1/2} \tag{1.2}$$

If $\varkappa > 1$, the equilibrium position $q_j = 0, p_j = 0$ is stable, while if $\varkappa < 1$, we have instability [2].

The case of $\varkappa = 1$ is critical. In the approximate system, the Hamiltonian of which is obtained from formula (1.1) by dropping terms higher than the fourth power in q_j, p_j, the equilibrium position is unstable. But it can be shown that terms whose powers are higher than the fourth can be chosen in such a way as to obtain stability or instability, as desired. For example, consider a system with a Hamilton function of the form

$$H = 3r_1 - r_2 + r_2^2 + \sqrt{3r_1 r_2}\, r_2 \sin\phi + a r_1^3 \tag{1.3}$$

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Here we have used the notation from (1.1), and a is constant coefficient. The value of $\varkappa$ for Hamiltonian (1.3) is equal to unity, i.e. we have the critical case.

A system with Hamiltonian (1.3) has two first integrals

$$V_1 = r_2 - 3r_1 = c = \text{const}, \quad V_2 = H = h = \text{const} \tag{1.4}$$

If $a = 1$, the equilibrium position is stable, which can be shown using Lyapunov's theorem on stability [1], taking the function $V = V_1^2 + V_2^2$ as Lyapunov's function. When $a = 1$ it will be positive-definite in r_1, r_2, whence the stability follows.

If $a = -1$, the equilibrium position will be unstable. To convince ourselves of this, consider the motion at the common levels $V_1 = 1$, $V_2 = 0$ of the first integrals (1.4). At these levels either $r_1 = r_2 = 0$ or $r_2 = 3r_1 = 27(1 + \sin\phi)$. The first case is of no interest to us since it corresponds to the equilibrium position itself. In the second case we have a particular solution which is doubly asymptotic to the point $r_1 = 0$. The trajectory corresponding to this solution is the cardioid $r_1 = 9(1 + \sin\phi)$. If $t \to \pm\infty$, then $r_1 \to 0$, while $\phi \to -\pi/2$. In this case the following equalities hold

$$\frac{d\phi}{dt} = -81(1+\sin\phi)^2, \quad \frac{dr_1}{dt} = -9r_1^2\cos\phi$$

If we put $\phi(0) = -\pi/2 - \mu$, where $0 < \mu \ll 1$, then $r_1(0) = 18\sin^2(\mu/2) \sim \mu^2$. The angle ϕ decreases monotonically with time, and as long as it remains in the third quadrant ($-\pi < \phi < -\pi/2$), the value of r_1 increases monotonically from as small a value as desired $r_1(0)$ to $r_1 = 9$, which also indicates that the equilibrium position is unstable.

This example shows that the problem of the stability of an equilibrium position at resonance $\omega_1 = 3\omega_2$ in the critical case, when $\varkappa = 1$, requires a special consideration which takes into account terms higher than the fourth power in expansion (1.1).

2. FORMULATION OF THE RESULT

Using a normalizing canonical transformation (obtained, for example, using the Deprit–Hori method [2]), the variables q_j, p_j can be chosen so that the Hamilton function $H(q_1, q_2, p_1, p_2)$ has a normal form not only in terms up to the fifth power, as in (1.1), but also in terms up to the seventh power inclusive. Calculations show that then

$$\begin{aligned} H &= \omega_1 r_1 - \omega_2 r_2 + c_{20}r_1^2 + c_{11}r_1r_2 + c_{02}r_2^2 + r_1^{1/2}r_2^{3/2}(a_{13}\sin\phi + b_{13}\cos\phi) + \\ &+ c_{30}r_1^3 + c_{21}r_1^2r_2 + c_{12}r_1r_2^2 + c_{03}r_2^3 + r_1^{3/2}r_2^{3/2}(a_{33}\sin\phi + b_{33}\cos\phi) + \\ &+ r_1^{1/2}r_2^{5/2}(a_{15}\sin\phi + b_{15}\cos\phi) + O_8 \end{aligned} \tag{2.1}$$

where we have used the notation from (1.1), and the set of terms whose powers in q_j, p_j is higher than the seventh, is denoted by O_8.

Theorem. If the coefficients of Hamilton function (2.1) are such that the quantity $\varkappa$, defined by (1.2), is equal to unity, but, in which case, the inequality

$$\begin{aligned} &(c_{20} + 3c_{11} + 9c_{02})(c_{30} + 3c_{21} + 9c_{12} + 27c_{03}) - \\ &- 27[a_{13}(a_{33} + 3a_{15}) + b_{13}(b_{33} + 3b_{15})] < 0 \end{aligned} \tag{2.2}$$

is satisfied, the equilibrium position $q_1 = q_2 = p_1 = p_2 = 0$ is unstable; for the opposite sign in inequality (2.2) we have stability.

3. PROOF OF THE THEOREM

The theorem is proved using the KAM-theory and the second Lyapunov method [1, 3], and in technical respects is based very much on the approach in [4], used when analysing the critical case of the problem of the stability of a Hamilton system with one degree of freedom with fourth-order resonance that is periodic with respect to the independent variable.

Instability. To prove the assertion on instability it is sufficient to show instability at the zeroth level of the energy integral H = const, on which the equilibrium position in question is situated. From the relation $H = 0$ we obtain $r_2 = -K(r_1, \varphi_1, \varphi_2)$. The motion along the isoenergy level $H = 0$ is described by Whittaker's equations [5], which have the form of Hamilton's equations, where the function K plays the role of the Hamilton function, while the independent variable is the quantity φ_2.

We will introduce the independent variable $\varphi_2^* = -\varphi_2$ instead of φ_2^*. The quantity φ_2^* in a small neighbourhood of the equilibrium position increases monotonically and can play the role of time in the stability problem. If we make the univalent canonical replacement of variables $r_1 = r_1^*/4$, $\varphi_1 = 4\varphi_1^* + 3\varphi_2^*$, a Hamiltonian of the following form will correspond to motion along the isoenergy level $H = 0$

$$K^* = \left[b_2 + 3\sqrt{3}\left(a_{13}^* \sin 4\varphi_1^* + b_{13}^* \cos 4\varphi_1^*\right)\right] r_1^{*2} + \left(b_3 + a_{33}^* \sin 4\varphi_1^* + b_{33}^* \cos 4\varphi_1^*\right) r_1^{*3} +$$
$$+\left(d_3 + d_{13}^* \sin 4\varphi_1^* + e_{13}^* \cos 4\varphi_1^*\right)\left[b_2 + 3\sqrt{3}\left(a_{13}^* \sin 4\varphi_1^* + b_{13}^* \cos 4\varphi_1^*\right)\right] r_1^{*3} + O\left(r_1^{*4}\right) \qquad (3.1)$$

Here

$$b_2 = (c_{20} + 3c_{11} + 9c_{02})/(16\omega_2), \quad a_{13}^* = a_{13}/(16\omega_2), \quad b_{13}^* = b_{13}/(16\omega_2)$$
$$b_3 = (c_{30} + 3c_{21} + 9c_{12} + 27c_{03})/(64\omega_2), \quad a_{33}^* = 3\sqrt{3}(a_{33} + 3a_{15})/(64\omega_2)$$
$$b_{33}^* = 3\sqrt{3}(b_{33} + 3b_{15})/(64\omega_2), \quad d_3 = (c_{11} + 6c_{02})/(4\omega_2) \qquad (3.2)$$
$$d_{13}^* = 3\sqrt{3}a_{13}/(8\omega_2), \quad e_{13}^* = 3\sqrt{3}b_{13}/(8\omega_2)$$

The structure of Hamiltonian (3.1) can be simplified somewhat by making a replacement of variables in accordance with the formulae

$$\varphi_1^* = \sigma(\varphi + \chi), \quad r_1^* = r, \quad \varphi_2^* = \delta\tau \qquad (3.3)$$

$$\delta = \left[27\left(a_{13}^{*2} + b_{13}^{*2}\right)\right]^{-1/2}, \quad \sin 4\chi = -3\sqrt{3}\delta a_{13}^*, \quad \cos 4\chi = -3\sqrt{3}\sigma\delta b_{13}^*, \quad \sigma = \operatorname{sign} b_2 \qquad (3.4)$$

In the new variables the motion is described by canonical equations with Hamiltonian

$$K = (1 - \cos 4\varphi)r^2 + (\gamma_3 + \gamma_{33}\sin 4\varphi + \delta_{33}\cos 4\varphi)r^3 +$$
$$+(d_3 + d_{13}\sin 4\varphi + e_{13}\cos 4\varphi)(1 - \cos 4\varphi)r^3 + O\left(r^4\right) \qquad (3.5)$$

$$\gamma_3 = \sigma\delta b_3, \quad \gamma_{33} = \delta\left(a_{33}^* \cos 4\chi - \sigma b_{33}^* \sin 4\chi\right), \quad \delta_{33} = \delta\left(a_{33}^* \sin 4\chi + \sigma b_{33}^* \cos 4\chi\right)$$
$$d_{13} = \sigma d_{13}^* \cos 4\chi - e_{13}^* \sin 4\chi, \quad e_{13} = \sigma d_{13}^* \sin 4\chi + e_{13}^* \cos 4\chi \qquad (3.6)$$

An even greater simplification of Hamiltonian (3.5) can be obtained using the replacement of variables

$$\varphi = \theta - \gamma_{33}(4|e|)^{-1}\rho, \quad r = |e|^{-1}\rho, \quad \tau = |e|\eta \quad (e = \gamma_3 + \delta_{33}) \qquad (3.7)$$

This replacement eliminates the term $r^3\gamma_{33}\sin 4\varphi$ in it and reduces the new Hamiltonian to the form

$$K = (1 - \cos 4\theta)\rho^2 + [s + d(1 - \cos 4\theta)]\rho^3 + O\left(\rho^4\right) \qquad (3.8)$$

where

$$s = \operatorname{sign} e, \quad d = (d_3 - \delta_{33} + d_{13}\sin 4\theta + e_{13}\cos 4\theta)|e|^{-1} \qquad (3.9)$$

If $s = -1$, instability occurs, which can be proved using Lyapunov's first theorem on instability [1]. The sign-variable function V can be taken [4] in the form $V = \rho^2 \sin 4\theta$. For sufficiently small ρ its derivative $dV/d\eta$ will be negative-definite [4] by virtue of the equations of motion with Hamiltonian (3.8). But it follows from formulae (3.2), (3.4), (3.6), (3.7) and (3.9) that the equality $s = -1$ is equivalent to condition (2.2). The assertion of the theorem on instability is proved.

Stability. Suppose now that $s = 1$, i.e. we have the opposite sign in inequality (2.2). In the system with Hamiltonian (2.1) we make the replacement (a canonical transformation with valency ε^{-1})

$$\varphi_1 = 4\varphi_1^* + 3\varphi_2^*, \quad \varphi_2 = -\varphi_2^*, \quad r_1 = \varepsilon r_1^*/4, \quad r_2 = \varepsilon(3r_1^* - 4r_2^*)/4, \quad t = t^*/\omega_2 \tag{3.10}$$

Since we are investigating motion in the neighbourhood of equilibrium, we have $0 < \varepsilon \ll 1$. The following Hamiltonian corresponds to the new variable

$$\begin{aligned} H^* = r_2^* + \varepsilon\Big\{\big[b_2 + 3\sqrt{3}\big(a_{13}^* \sin 4\varphi_1^* + b_{13}^* \cos 4\varphi_1^*\big)\big] r_1^{*2} + f_1\big(r_1^*, r_2^*, \varphi_1\big) + \\ + \varepsilon\big[\big(b_3 + a_{33}^* \sin 4\varphi_1^* + b_{33}^* \cos 4\varphi_1^*\big) r_1^{*3} + f_2\big(r_1^*, r_2^*, \varphi_1\big)\big]\Big\} + \varepsilon^3 f_3\big(r_1^*, r_2^*, \varphi_1^*, \varphi_2^*; \varepsilon^{1/2}\big) \end{aligned} \tag{3.11}$$

The coefficients b_2, a_{13}^*, b_{13}^*, b_3, a_{33}^*, b_{33}^* are calculated from formulae (3.2). The last term in (3.11) is the set of terms O_8 from (2.1), transformed to the new variable. The explicit form of the functions f_1 and f_2 in expression (3.11) is not required; it is only important that these functions should be independent of the variable φ_2^*, vanish when $r_2^* = 0$ and be analytical with respect to their arguments r_1^*, r_2^*, φ_1^* when $r_1^* > 0$. Note also that the variable $r_1^* \geq 0$, while r_2^* can take values of any sign.

One more canonical transformation (with valency σ)

$$\varphi_1^* = \sigma(\theta_1^* + \chi), \quad \varphi_2^* = \theta_2^*, \quad r_1^* = \rho_1^*, \quad r_2^* = \sigma\rho_2^*, \quad t^* = \delta\tau^* \tag{3.12}$$

where σ, χ, δ are defined by (3.4), leads to equations of motion with Hamiltonian

$$\begin{aligned} \Gamma^* = \delta\rho_2^* + \varepsilon\Big\{\big(1 - \cos 4\theta_1^*\big)\rho_1^{*2} + f_1^*\big(\rho_1^*, \rho_2^*, \theta_1^*\big) + \varepsilon\big[\big(\gamma_3 + \gamma_{33} \sin 4\theta_1^* + \delta_{33} \cos 4\theta_1^*\big)\rho_1^{*3} + \\ + f_2^*\big(\rho_1^*, \rho_2^*, \theta_1^*\big)\big]\Big\} + \varepsilon^3 f_3^*\big(\rho_1^*, \rho_2^*, \theta_1^*, \theta_2^*; \varepsilon^{1/2}\big) \end{aligned} \tag{3.13}$$

where f_i^* is the function f_i from (3.11) multiplied by $\sigma\delta$, in which the arguments are expressed in terms of the new variables from formulae (3.12) while the coefficients γ_3, γ_{33}, δ_{33} are defined by (3.6).

Finally, we make a replacement of variables similar to (3.7)

$$\theta_1^* = \theta - \varepsilon\gamma_{33}(4|e|)^{-1}\rho, \quad \rho_1^* = |e|^{-1}\rho, \quad \theta_2^* = w_2, \quad \rho_2^* = |e|^{-1} I_2, \quad \tau^* = |e|\eta$$

In the new variables the motion is described by canonical equations with Hamiltonian

$$F = F^{(0)}(I_2) + \varepsilon F^{(1)}(\rho, I_2, \theta; \varepsilon) + \varepsilon^3 F^{(2)}\big(\rho, I_2, \theta, w_2; \varepsilon^{1/2}\big) \tag{3.14}$$

where

$$F^{(0)} = \delta|e|I_2, \quad F^{(1)} = (1 - \cos 4\theta)\rho^2 + g_1(\rho, I_2, \theta) + \varepsilon\{[s + k(1 - \cos 4\theta)]\rho^3 + g_2(\rho, I_2, \theta)\}$$

$$s = \operatorname{sign} e = 1, \quad k = -\delta_{33}|e|^{-1}$$

The function F is 2π-periodic in the variables θ, w_2 and for $\rho > 0$ and sufficiently small ε is analytical in ρ, I_2, θ, w_2, $\varepsilon^{1/2}$. The functions g_1 and g_2 vanish when $I_2 = 0$.

If we drop the last term in Hamiltonian (3.14), we obtain an approximate system with Hamiltonian $F^{(0)} + \varepsilon F^{(1)}$. It has two first integrals $F^{(0)} + \varepsilon F^{(1)} = h = \text{const}$ and $I_2 = I_2(0) = \text{const}$ and is integrable. Suppose $I_2(0)$ is a small quantity (for example, of the order of $\varepsilon^{5/2}$). Then, for sufficiently small values of

h, the trajectories of the approximate system in the $x_1 = \sqrt{2\rho}\cos\theta, x_2 = \sqrt{2\rho}\sin\theta$ plane will be closed, encircling the point $x_1 = x_2 = 0$ (see [4, Fig. 3b]). For the corresponding motions of the approximate system we can introduce action-angle variables I_j, w_j $(j = 1, 2)$. In view of the fact that, in the approximate system, the coordinate w_2 is cyclical, one of the two pairs of these variables will be the pair I_2, w_2. We will denote the Hamilton function (3.14), written in the variables I_j, w_j $(j = 1, 2)$ by Φ.

$$\Phi = \Phi^{(0)}(I_2) + \varepsilon\Phi^{(1)}(I_1, I_2) + \varepsilon^3\Phi^{(2)}\left(I_1, I_2, w_1, w_2; \varepsilon^{1/2}\right) \tag{3.15}$$

Here $\Phi^{(0)}$ is the function $F^{(0)}$ from (3.14). The function Φ is 2π-periodic in w_1 and w_2, and is analytic in I_1, I_2, w_1, w_2 and $\varepsilon^{1/2}$ when $I_1 > 0$.

If the following inequalities are satisfied

$$\frac{\partial\Phi^{(0)}}{\partial I_2} \neq 0, \quad \frac{\partial\Phi^{(1)}}{\partial I_1} \neq 0, \quad \frac{\partial^2\Phi^{(1)}}{\partial I_1^2} \neq 0 \tag{3.16}$$

then, according to the KAM-theory [3], the variables I_1, I_2 for all $\eta > 0$ remain in the region of their initial values: $|I_j(\eta) - I_j(0)| < c\varepsilon^2$ $(c = \text{const} > 0)$. Hence, to prove the assertion of the theorem on stability it is sufficient to show that inequalities (3.16) hold. The first of these is obviously satisfied. It was shown previously in [4] that the second and third hold when $I_2 = 0$. But in view of the fact that the function $\Phi^{(1)}$ is analytical, the last two inequalities will also hold for sufficiently small values of $|I_2|$. The theorem is proved.

4. THE STABILITY OF STEADY ROTATION OF AN ARTIFICIAL SATELLITE

Consider the motion of a dynamically symmetrical artificial satellite – of a rigid body about a centre of mass in a central Newtonian gravity field. It is well known [6], that in a circular orbit the satellite can move so that its axis of symmetry is always perpendicular to the plane of the orbit, while the satellite itself rotates around the axis of symmetry with an arbitrary constant angular velocity (cylindrical precession). The problem of the stability of the cylindrical precession has been investigated in some detail in [7–10]. The results so far obtained are as follows.

The motion of the axis of symmetry of the satellite is described by an autonomous canonical system of fourth-order differential equations, which depends on two parameters α and β $(\alpha = C/A, \beta = r_0/\omega_0$, where C and A are polar and equatorial moments of inertia, r_0 is the projection of the absolute angular velocity of the satellite on its axis of symmetry, which is the integral of motion, and ω_0 is the angular velocity of motion of the centre of mass in the orbit, $0 < \alpha \leqslant 2, -\infty < \beta < \infty)$. The characteristic equation of the linearized system of equations of perturbed motion is written in the form

$$\lambda^4 + a\lambda^2 + b = 0 \tag{4.1}$$

$$a = \alpha^2\beta^2 - 2\alpha\beta + 3\alpha - 1, \quad b = (\alpha\beta - 1)(\alpha\beta + 3\alpha - 4)$$

In regions defined by the inequalities

$$a > 0, \quad b > 0, \quad a^2 - 4b > 0 \tag{4.2}$$

there is stability, to a first approximation. In these regions, Eq. (4.1) has imaginary roots. If at least one of the inequalities (4.2) is satisfied with the opposite sign, then, in the characteristic equation, there is a root with positive real part, and by Lyapunov's theorem on stability in the first approximation [1], the cylindrical precession is unstable. In the figure the regions of instability are shown hatched in the plane of the parameters α, β. In the unhatched regions 1 and 2 the roots of the characteristic equation are pure imaginary.

In region 1 the cylindrical precession is stable [7]. In this region the Hamiltonian H is a sign-definite function of the coordinates and moments q_j, p_j $(j = 1, 2)$ of the perturbed motion (in q_j, p_j variables the cylindrical precession corresponds to the equilibrium position $q_j = p_j = 0$ on the axis of symmetry of the satellite in an orbital system of coordinates). This enabled us to use Lyapunov's second method to solve the stability problem. In region 2 the function H is not sign-definite (although stability occurs

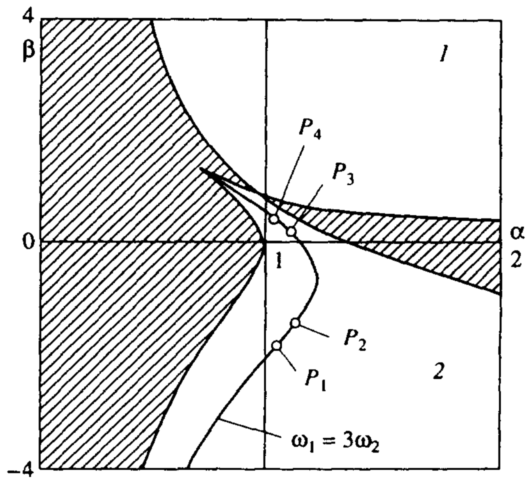

Fig. 1

in the first approximation), and, to solve the problem of the stability of the cylindrical precession of the satellite, it was necessary to make use of modern methods of the theory of the stability of Hamiltonian systems [2, 11, 12]. Calculations were carried out for $\beta \geqslant -20$. The problem of the stability of the cylindrical precession was solved (for $\beta \geqslant -20$) for all points $P(\alpha, \beta)$ from region 2, except five points: P_* and P_k ($k = 1, 2, 3, 4$). At the point P_* (1.064, 0.425) the Hamiltonian of the perturbed motion H is isoenergically degenerate (see below) when terms up to the sixth power inclusive in q_j, p_j are taken into account in its expansion in series. To solve the stability problem at this point one must take into account terms not lower than the eighth power. The points P_k ($k = 1, 2, 3, 4$) lie on the curve of fourth-order resonance $\omega_1 = 3\omega_2$. At each of these the quantity $\varkappa$, defined by (1.2), is equal to unity. Over the whole resonance curve the cylindrical precession is stable, with the exception of two parts of it P_1, P_2 and $P_3\,P_4$, on which there is instability. The boundary points P_k of the stability and instability sections are

$$P_1(1.052, -1.742),$$
$$P_2(1.087, -1.567),$$
$$P_3(1.072, 0.385), \quad P_4(1.056, 0.449)$$

For these points the stability problem is solved in this paper using the theorem of Section 2.

But initially we will consider the point P_*. At this point $\omega_1 = 1.61$ and $\omega_2 = 0.380$, and hence there is no resonance up to the eighth order inclusive (i.e. $k_1\omega_1 \neq k_2\omega_2$ for natural k_1 and k_2 which satisfy the inequalities $0 < k_1 + k_2 \leqslant 8$). Using a normalizing canonical transformation, obtained by the Deprit–Hori method [2], the variables q_j, p_j were chosen so that the Hamiltonian of the perturbed motion took the normal form up to terms of the eighth power inclusive

$$H = \omega_1 r_1 - \omega_2 r_2 + \sum_{k+l=2}^{4} c_{kl} r_1^k r_2^l + O\left((r_1 + r_2)^5\right) \qquad (4.3)$$

$$q_j = \sqrt{2r_j}\sin\varphi_j, \quad p_j = \sqrt{2r_j}\cos\varphi_j$$

where c_{kl} are numerical coefficients. The Hamiltonian was reduced to normal form (4.3) on a computer in the MAPLE V system.

We will introduce the notation

$$D_{2m} = \sum_{i=0}^{m} c_{m-i,i}\,\omega_1^i \omega_2^{m-i}$$

At the point P_* the quantities D_4 and D_6 are equal to zero [8, 10], i.e. the Hamiltonian of the perturbed motion is isoenergetically degenerate up to terms of the sixth power inclusive. In terms of the eighth power the degeneracy is removed, since calculations show that $D_8 = 4.48 \neq 0$. Consequently [2, 11, 12], the cylindrical precession of the satellite is stable for values of α and β corresponding to the point P_*.

To solve the problem of stability at the points P_k ($k = 1, 2, 3, 4$), which lie on the resonance curve $\omega_1 = 3\omega_2$, it turned out to be sufficient to consider terms up to the sixth power in the expansion of the Hamiltonian of the perturbed motion. At these points the Hamiltonian was reduced to normal form (2.1) using the Deprit–Hori method in the MAPLE V system. We will denote by Δ_k the value of the left-hand side of inequality (2.2) at the point P_k. Calculations show that

$$\Delta_1 = 0.076, \quad \Delta_2 = -0.097, \quad \Delta_3 = -3.158, \quad \Delta_4 = 5.475$$

Hence, by the theorem in Section 2, at points P_1 and P_4 the cylindrical precession of the satellite is stable, while at points P_2 and P_3 it is unstable.

Hence, the problem of the stability of the cylindrical precession of a satellite for values of the parameters α and β lying inside region 2 and which satisfy the inequality $\beta \geqslant -20$ (calculations were only carried out for these values of β), is completely solved. It is unstable along the two sections P_1P_2 and P_3P_4 of the resonance curve $\omega_1 = 3\omega_2$ and at two of their boundary points P_2 and P_3, and it is stable for the remaining values of the parameters α and β.

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